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An extension of the divergence operator for Gaussian processes

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Abstract

We extend the domain of the divergence operator δ for Gaussian processes in the sense of the calculus of variations. As an example, we discuss the case of the fractional Brownian motion with Hurst parameter in $(0, \frac{1}{2})$ defined on a finite time interval. If $H < \frac{1}{4}$ this process does not belong to the domain of δ, but it is in the extended domain.

Keywords: Gaussian processes on Hilbert spaces; Malliavin calculus; Fractional calculus; Divergence operator; Stochastic integral

1. Introduction

It is well known that the Itô stochastic integral of square integrable non-anticipating processes with respect to the Brownian motion coincides with the divergence operator δ, that is, the adjoint of the derivative operator in the Malliavin calculus (see [13] and the references therein). The Malliavin calculus can be developed with respect to any Gaussian process, and in this sense, a natural question

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is to interpret the associated divergence operator as a stochastic integral, and to develop a stochastic calculus. A particular example of a Gaussian process where the divergence operator has been extensively studied is the fractional Brownian motion.

The fractional Brownian motion (fBm) $\{B_t^H,\ t \geqslant 0\}$ with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process such that $B_0^H = 0$, and $E[(B_t^H - B_s^H)^2] = (t-s)^{2H}$ for all $0 \leqslant s < t$. The case $H = \frac{1}{2}$ corresponds to the Brownian motion. Since the fBm is not a semimartingale for $H \neq \frac{1}{2}$, the Itô approach to the construction of a stochastic integral with respect to fBm is not valid. Two main approaches have been suggested in order to define stochastic integrals with respect to fBm:

(i) A pathwise method using Young's integral [18] can be used in the case $H > \frac{1}{2}$. The theory of *rough path analysis* introduced by Lyons (see [12]) provides a pathwise approach to stochastic integration and stochastic differential equations with respect to the fBm in the case $H > \frac{1}{4}$ (see [9]). An alternative pathwise method based on fractional calculus has been introduced by Zähle [19].

(ii) Starting from the work by Decreusefond and Üstünel [10], several authors have developed a stochastic calculus with respect to the fBm using the divergence operator of Malliavin calculus (see [1–4,7,11,14], among others). This divergence-type integral has zero mean and can be obtained as the limit of Riemann sums using Wick products. Moreover, the divergence integral, turns out to be equal to the pathwise integral plus a complementary term (see [1,4]).

The drawback of these works on the divergence integral is that the process B^H does not belongs to the domain of the divergence operator when $H \leqslant \frac{1}{4}$. This was proved by Cheridito and Nualart [8], and in this paper these authors extended the divergence operator and obtained an Itô and Tanaka formulas for any value of H in $(0,1)$. The extended divergence is defined in [8] using integration by parts and a suitable family of test random variables. There is also another approach to define the divergence integral and to establish an Itô's formula for any value of H, using the techniques of white noise analysis (see [6,5]).

The aim of this paper is to generalize the construction of the extended divergence to any Gaussian process, and, on the other hand, to characterize its domain using the Wiener chaos expansion (Theorem 3.2). The paper is organized as follows: Section 2 contains the basic definitions of the derivative and divergence operators in the framework of a general Gaussian process. Notice that we are interested in random variables taking values in a Hilbert space which is larger than the underlying Hilbert space associated with the Gaussian process. In Section 3, we introduce the extended divergence operator and we prove its characterization in terms of the chaos decomposition. In Section 4, we consider the particular case of the fBm and we recover the results by Cheridito and Nualart [8].

2. Preliminaries

In this section, we describe the basic framework that is used in this article. For this, we assume that the reader is familiar with both the elementary facts of the

Malliavin calculus (as given, e.g. in Nualart [13]), as well as the theory of tensor products of Hilbert spaces (as presented, e.g. in Reed and Simon [16]).

Let $\mathscr{H}$ and $\mathscr{H}_0$ be two real separable Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ and $\langle\cdot,\cdot\rangle_{\mathscr{H}_0}$, respectively. Throughout we assume that $\mathscr{H}$ is densely and continuously embedded in $\mathscr{H}_0$ and that $T:\mathscr{H}\subset\mathscr{H}_0\to\mathscr{H}_0$ is a linear operator (whose domain $\mathscr{D}(T)$ is $\mathscr{H}$) satisfying the following conditions:

(H1) $|Th|_{\mathscr{H}_0}=|h|_{\mathscr{H}}$ for all $h\in\mathscr{H}$.
(H2) $\mathscr{T}_{\mathscr{H}}:=\{h\in\mathscr{H}:Th\in\mathscr{D}(T^*)\}$ is a dense subset of $\mathscr{H}$.
(H3) $\mathscr{T}_{\mathscr{H}_0}=\{T^*Th:h\in\mathscr{T}_{\mathscr{H}}\}$ is dense in $\mathscr{H}_0$.

Note that (H1) gives that T is a closed operator on $\mathscr{H}_0$. Therefore $\mathscr{D}(T^*)$ is also a dense subset of $\mathscr{H}_0$ (see [16, Theorem VIII.1]).

Now we state the following result, which is used in Section 3.

Lemma 2.1. *Let* $n\geqslant 1$, $f\in\mathscr{H}^{\otimes n}\otimes\mathscr{H}_0$, $h\in\mathscr{H}_0$ *and* $\{g_1,\dots,g_n\}\subset\mathscr{T}_{\mathscr{H}}$. *Then* (H1) *implies that*

$$\langle f,g_1\otimes\cdots\otimes g_n\otimes h\rangle_{\mathscr{H}^{\otimes n}\otimes\mathscr{H}_0}=\langle f,T^*Tg_1\otimes\cdots\otimes T^*Tg_n\otimes h\rangle_{\mathscr{H}_0^{\otimes(n+1)}}$$

holds.

Proof. Let $\{h_i\}$ and $\{k_i\}$ be orthonormal bases of $\mathscr{H}^{\otimes n}$ and $\mathscr{H}_0$, respectively. Then

$$f=\sum_{i,j}a_{ij}h_i\otimes k_j\quad\text{in }\mathscr{H}^{\otimes n}\otimes\mathscr{H}_0,$$

for some sequence $(a_{ij})_{i,j\in\mathbb{N}}\subset\mathbb{R}$.

Now by Hypothesis (H1) and [16, Section VIII.10], we have for each $m\in\mathbb{N}$,

$$\begin{aligned}&\left\langle\sum_{i,j=1}^m a_{ij}h_i\otimes k_j,g_1\otimes\cdots\otimes g_n\otimes h\right\rangle_{\mathscr{H}^{\otimes n}\otimes\mathscr{H}_0}\\&\quad=\sum_{i,j=1}^m a_{ij}\langle T^{\otimes n}(h_i),T^{\otimes n}(g_1\otimes\cdots\otimes g_n)\rangle_{\mathscr{H}_0^{\otimes n}}\langle k_j,h\rangle_{\mathscr{H}_0}\\&\quad=\left\langle T^{\otimes n}\otimes I_{\mathscr{H}_0}\left(\sum_{i,j=1}^m a_{ij}h_i\otimes k_j\right),Tg_1\otimes\cdots\otimes Tg_n\otimes h\right\rangle_{\mathscr{H}_0^{\otimes(n+1)}},\end{aligned}$$

where $I_{\mathscr{H}_0}$ is the identity operator on $\mathscr{H}_0$. Letting m tend to infinity we obtain

$$\langle f,g_1\otimes\cdots\otimes g_n\otimes h\rangle_{\mathscr{H}^{\otimes n}\otimes\mathscr{H}_0}=\langle T^{\otimes n}\otimes I_{\mathscr{H}_0}(f),Tg_1\otimes\cdots\otimes Tg_n\otimes h\rangle_{\mathscr{H}_0^{\otimes(n+1)}}.$$

Thus the proof is complete. $\square$

2.1. The chaos decomposition

Henceforth $W=\{W(h):h\in\mathscr{H}\}$ is an *isonormal* Gaussian process on $\mathscr{H}$ defined in a complete probability space $(\Omega,\mathscr{F},P)$. That is, W is a family of centered Gaussian random variables such that $E(W(h)W(g)) = \langle h, g\rangle_{\mathscr{H}}$, for $h, g \in \mathscr{H}$. We will assume that $\mathscr{F}$ is the σ-field generated by W.

It is well known (see [13]) that $L^2(\Omega; \mathscr{H}_0)$ agrees with the space $\bigoplus_{n=0}^{\infty} \mathscr{H}^{\odot n} \otimes \mathscr{H}_0$, where $\mathscr{H}^{\odot 0} = \mathbb{R}$ and $\mathscr{H}^{\odot n}$ is the nth symmetric tensor product of $\mathscr{H}$. Hence we can see that each $F \in L^2(\Omega; \mathscr{H}_0)$ has a unique chaos representation of the form

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where for each $n \geqslant 0$, $f_n \in \mathscr{H}^{\odot n} \otimes \mathscr{H}_0$ and the $\mathscr{H}_0$-valued square integrable random variable $I_n(f_n)$ is characterized by

$$E(\langle h, I_n(f_n)\rangle_{\mathscr{H}_0}(n_{i_1})!H_{n_{i_1}}(W(e_{i_1}))\cdots(n_{i_k})!H_{n_{i_k}}(W(e_{i_k}))) = \begin{cases} n!\langle f_n, e_{i_1}^{\otimes n_{i_1}} \otimes \cdots \otimes e_{i_k}^{\otimes n_{i_k}} \otimes h\rangle_{\mathscr{H}^{\otimes n}\otimes \mathscr{H}_0} & \text{if } \sum_{j=1}^{k} n_{i_j} = n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Here $h \in \mathscr{H}_0$, $\{e_i : i \in \mathbb{N}\}$ is an orthonormal basis of $\mathscr{H}$ and from now on, H_n denotes the nth Hermite polynomial defined by

$$H_n(x) = \frac{(-1)^n}{n!} \mathrm{e}^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x^2/2}), \quad x \in \mathbb{R} \text{ and } n \geqslant 0.$$

2.2. The derivative operator

Let $C_p^{\infty}(\mathbb{R}^n)$ be the set of C^{∞} functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all its partial derivatives have polynomial growth. For a real separable Hilbert space $\mathscr{K}$, $I_{\mathscr{K}}$ denotes the identity operator on $\mathscr{K}$, $(T \otimes I_{\mathscr{K}})^*$ is the adjoint of the operator $T \otimes I_{\mathscr{K}} : \mathscr{H} \otimes \mathscr{K} \subset \mathscr{H}_0 \otimes \mathscr{K} \to \mathscr{H}_0 \otimes \mathscr{K}$ and $\mathscr{S}(K)$ (resp. $\mathscr{S}_T(\mathscr{K})$) represents the class of smooth random variables of the form

$$F = f(W(g_1), \ldots, W(g_n))k, \tag{2.2}$$

where $k \in \mathscr{K}$, $\{g_1, \ldots, g_n\}$ is in $\mathscr{H}$ (resp. in $\mathscr{T}_{\mathscr{H}}$) and $f \in C_p^{\infty}(\mathbb{R}^n)$.

The derivative of the smooth random variable F given by (2.2) is the $\mathscr{H} \otimes \mathscr{K}$-valued random variable DF defined by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(g_1), \ldots, W(g_n))g_i \otimes k.$$

It is well-known that D is a closable operator from $L^2(\Omega; \mathscr{K})$ into $L^2(\Omega; \mathscr{H} \otimes \mathscr{K})$. The domain $\mathbb{D}^{1,2}(\mathscr{K})$ of the closure of D in $L^2(\Omega; \mathscr{K})$ (also denoted by D) is the completion of $\mathscr{S}(\mathscr{K})$ with respect to the norm

$$||F||_{1,2}^2 = E(|F|_{\mathscr{K}}^2 + |DF|_{\mathscr{H}\otimes\mathscr{K}}^2).$$

In this paper we also consider the operator

$$D_T : \mathscr{S}_T(\mathscr{K}) \subset L^2(\Omega; \mathscr{K}) \to L^2(\Omega; \mathscr{H}_0 \otimes \mathscr{K})$$

defined by

$$D_T(F) = (T \otimes I_{\mathscr{K}})^*(T \otimes I_{\mathscr{K}})DF, \quad F \in \mathscr{S}_T(\mathscr{K}). \tag{2.3}$$

In the appendix of this article (see Section 5 for details), we show that this operator is also closable from $L^2(\Omega; \mathscr{K})$ into $L^2(\Omega; \mathscr{H}_0 \otimes \mathscr{K})$. The domain of its closure (also denoted by D_T) in $L^2(\Omega; \mathscr{K})$ is the set $\mathbb{D}_T^{1,2}(\mathscr{K})$. It means that $\mathbb{D}_T^{1,2}(\mathscr{K})$ is the completion of the $\mathscr{K}$-valued smooth random variables $\mathscr{S}_T(\mathscr{K})$ with respect to the norm

$$||F||_{1,2,T}^2 = E(|F|_{\mathscr{K}}^2 + |(T \otimes I_{\mathscr{K}})^*(T \otimes I_{\mathscr{K}})DF|_{\mathscr{H}_0 \otimes \mathscr{K}}^2).$$

We also prove in Section 5 that if $F \in \mathbb{D}_T^{1,2}(\mathscr{K})$, then we have $(T \otimes I_{\mathscr{K}})DF$ belongs to $\mathrm{Dom}((T \otimes I_{\mathscr{K}})^*)$ w.p.1, and

$$D_T F = (T \otimes I_{\mathscr{K}})^*(T \otimes I_{\mathscr{K}})DF.$$

3. The divergence operator

Recall that $\mathscr{H}$ is densely and continuously embedded in $\mathscr{H}_0$ and that $T : \mathscr{H} \subset \mathscr{H}_0 \to \mathscr{H}_0$ is a linear operator satisfying hypotheses (H1)–(H3).

Definition 3.1. Let $u \in L^2(\Omega; \mathscr{H}_0)$. We say that u belongs to $\mathrm{Dom}^*\delta$ if and only if there exists $\delta(u) \in L^2(\Omega)$ such that

$$E\langle D_T F, u\rangle_{\mathscr{H}_0} = E(F\delta(u)) \quad \text{for every } F \in \mathscr{S}_T(\mathbb{R}). \tag{3.1}$$

In this case, the random variable $\delta(u)$ is called the extended divergence of u.

Remarks.

(i) Hypothesis (H2) yields that there is at most one square integrable random variable $\delta(u)$ such that (3.1) holds.
(ii) If $\mathscr{H}_0 = \mathscr{H}$ and $T = I_{\mathscr{H}}$, then δ is equal to the usual divergence operator presented in [13].
(iii) Observe that the duality relation (3.1) also holds for $F \in \mathbb{D}_T^{1,2}(\mathbb{R})$.

The following result characterizes the set $\mathrm{Dom}^*\delta$.

Theorem 3.2. *Assume that* (H1)–(H3) *hold and that* $u \in L^2(\Omega; \mathscr{H}_0)$ *has the chaos representation*

$$u = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in \mathscr{H}^{\odot n} \otimes \mathscr{H}_0.$$

Then $u \in \mathrm{Dom}^*\delta$ *if and only if* $\widetilde{f}_n$ *(symmetrization of* f_n *as an element of* $\mathscr{H}_0^{\otimes(n+1)}$*) belongs to* $\mathscr{H}^{\odot(n+1)}$ *for all* $n \geqslant 0$, *and*

$$\sum_{n=1}^{\infty} n! \left| \widetilde{f}_{n-1} \right|^2_{\mathscr{H}^{\otimes n}} < \infty. \tag{3.2}$$

In this case $\delta(u) = \sum_{n=1}^{\infty} I_n(\widetilde{f}_{n-1})$.

Proof. Fix $n \geqslant 1$. Let $\{n_1, \ldots, n_k\}$ be a finite sequence of positive integers such that $n_1 + \cdots + n_k = n$ and $\{g_1, \ldots, g_k\} \subset \mathscr{T}_{\mathscr{H}}$ an orthonormal system on $\mathscr{H}$.

Necessity: From Lemma 2.1 and (2.1), we have

$$\begin{aligned} &E\langle u, D_T(n_1! H_{n_1}(W(g_1)) \cdots n_k! H_{n_k}(W(g_k)))\rangle_{\mathscr{H}_0} \\ &\quad = \sum_{j=1}^{k} n_j (n-1)! \langle f_{n-1}, (T^*T)^{\otimes(n-1)} \\ &\qquad \times (g_1^{\otimes n_1} \otimes \cdots \otimes g_{j-1}^{\otimes n_{j-1}} \otimes g_j^{\otimes(n_j-1)} \otimes \cdots \otimes g_{n_k}^{\otimes n_k}) \otimes T^*Tg_j\rangle_{\mathscr{H}_0^{\otimes n}}. \end{aligned}$$

Hence, if $\delta(u)$ has the chaos representation

$$\delta(u) = \sum_{n=0}^{\infty} I_n(v_n), \quad v_n \in \mathscr{H}^{\odot n},$$

then the duality relation (3.1) and (H3) yield that $v_n = \widetilde{f}_{n-1}$, and therefore (3.2) is true.

Sufficiency: Let $F = f(W(g_1), \ldots, W(g_k))$ be a random variable in $\mathscr{S}_T(\mathbb{R})$ and $\mathscr{K}$ the linear subspace of $\mathscr{H}$ generated by $\{g_1, \ldots, g_k\}$. Then by Section 2.1, F has the chaos decomposition given by

$$F = \sum_{n=0}^{\infty} I_n(k_n), \quad k_n \in \mathscr{K}^{\odot n}.$$

Consequently, using Lemma 2.1 again, we obtain

$$\begin{aligned} E\langle u, D_T F\rangle_{\mathscr{H}_0} &= \sum_{n=0}^{\infty} (n+1)! \langle f_n, (T^*T)^{\otimes(n+1)}(k_{n+1})\rangle_{\mathscr{H}_0^{\otimes(n+1)}} \\ &= \sum_{n=0}^{\infty} (n+1)! \left\langle \widetilde{f}_n, (T^*T)^{\otimes(n+1)}(k_{n+1}) \right\rangle_{\mathscr{H}_0^{\otimes(n+1)}} \\ &= \sum_{n=0}^{\infty} (n+1)! \left\langle \widetilde{f}_n, k_{n+1} \right\rangle_{\mathscr{H}^{\otimes(n+1)}} \\ &= E\left(F \sum_{n=1}^{\infty} I_n\left(\widetilde{f}_{n-1}\right)\right). \end{aligned}$$

That is, the duality relation (3.1) is satisfied for u and $\delta(u) := \sum_{n=1}^{\infty} I_n(\widetilde{f}_{n-1})$, in this case. So the proof is complete. $\square$

The following result shows that the operator δ is local in $\mathbb{D}_T^{1,2}(\mathscr{H}_0)$.

Proposition 3.3. *Let $u \in \mathrm{Dom}^*\delta \cap \mathbb{D}_T^{1,2}(\mathscr{H}_0)$ and $A \in \mathscr{F}$ be such that $u = 0$ on A. Then $\delta(u) = 0$ on A w.p.1.*

Proof. As in [13, proof of Proposition 1.3.6], $\phi : \mathbb{R} \to \mathbb{R}_+$ is a C^∞ function such that $\phi(0) = 1$ and its support is included in $[-1, 1]$. For $\varepsilon > 0$, set $\phi_\varepsilon(x) = \phi(x/\varepsilon)$. Now, let $F = f(W(g_1), \ldots, W(g_n))$ be in $\mathscr{S}_T(\mathbb{R})$ with $f \in C_0^\infty(\mathbb{R}^n)$ (i.e., f has compact support). Then Lemma A.4 below and the duality relation (3.1) give

$$\begin{aligned} &E[\delta(u)\phi_\varepsilon(|u|^2_{\mathscr{H}_0})F] \\ &\quad = E\langle D_T\phi_\varepsilon(|u|^2_{\mathscr{H}_0}), Fu\rangle_{\mathscr{H}_0} + E\langle D_T F, \phi_\varepsilon(|u|^2_{\mathscr{H}_0})u\rangle_{\mathscr{H}_0} \\ &\quad = E\{2F\langle ((T \otimes I_{\mathscr{H}_0})^*(T \otimes I_{\mathscr{H}_0})Du)^*(u), \phi'_\varepsilon(|u|^2_{\mathscr{H}_0})u\rangle_{\mathscr{H}_0} \\ &\qquad + \langle D_T F, \phi_\varepsilon(|u|^2_{\mathscr{H}_0})u\rangle_{\mathscr{H}_0}\}. \end{aligned}$$

Finally, proceeding as in [13, proof of Proposition 1.3.6] we obtain

$$\delta(u)1_{\{|u|^2_{\mathscr{H}_0}=0\}} = 0 \quad \text{w.p.1}$$

and therefore the proof is finished. □

As an immediate consequence of Proposition 3.3 we can extend (or localize) the domain of δ as follows. We say that $u \in (\mathrm{Dom}^*\delta)_{\mathrm{loc}}$ if there exists a sequence $\{(\Omega_n, u^{(n)}) : n \geqslant 1\} \subset \mathscr{F} \times \mathbb{D}_T^{1,2}(\mathscr{H}_0)$ such that $\Omega_n \uparrow \Omega$ and $u = u^{(n)}$ on Ω_n w.p.1. In this case we define

$$\delta(u)|_{\Omega_n} = \delta(u^{(n)}), \quad n \geqslant 1.$$

4. An example

Throughout this section, $a \in (0, \infty)$ and $B^H = \{B_t^H : t \in [0, a]\}$ is a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, \frac{1}{2})$. Here we show that there is an operator T satisfying Hypotheses (H1–H3) such that $B^H \in (\mathrm{Dom}^*\delta \backslash \mathrm{Dom}\delta)$ whether $H \in (0, \frac{1}{4})$, where $\mathrm{Dom}\delta$ is the domain of the (usual) divergence operator.

From Pipiras and Taqqu [15, Theorem 4.2], we know that $B^H(f) = \int_0^a f_t \, \mathrm{d}B_t^H$ defines an isonormal Gaussian process on the Hilbert space

$$\begin{aligned} \mathscr{H} = \Lambda_T^\alpha := \{f : [0, a] \to \mathbb{R} : \exists \phi_f \in L^2([0, a]) \\ \text{such that } f(u) = u^\alpha(I_{a-}^\alpha(s^{-\alpha}\phi_f(s)))(u)\} \end{aligned}$$

equipped with the inner product

$$\langle f, g\rangle_{\mathscr{H}} = C_H\langle \phi_f, \phi_g\rangle_{L^2([0,a])}. \tag{4.1}$$

Here $\alpha = \frac{1}{2} - H$, $C_H = \pi\alpha(2\alpha - 1)(\Gamma(1 + 2\alpha)\sin(-\pi\alpha))^{-1}$ and I_{a-}^α is the *right-sided fractional Riemann–Liouville integral of order* α (see Samko et al. [17] for a complete treatment of the fractional calculus). This fractional integral is defined for integrable functions f on $[0,a]$. It is given by

$$(I^{\alpha}_{a-}f)(s) = \Gamma(\alpha)^{-1}\int_s^a f(u)(u-s)^{\alpha-1}\,\mathrm{d}u \quad \text{for a.a. } s \in [0,a].$$

We denote by D^{α}_{a-} the *right-sided fractional derivative* or order α, which is the inverse of I^{α}_{a-}.

The following result yields that $L^2([0,a])$ plays the role of the space $\mathscr{H}_0$.

Proposition 4.1. *The space $\mathscr{H}$ is densely and continuously embedded in $L^2([0,a])$.*

Proof. Let $f \in \mathscr{H}$. Then there exists $\phi_f \in L^2([0,a])$ such that

$$\begin{aligned}\int_0^a (f(u))^2\,\mathrm{d}u &\leqslant C_\alpha \int_0^a \left(\int_u^a (r-u)^{\alpha-1}\phi_f(r)\,\mathrm{d}r\right)^2 \mathrm{d}u \\ &\leqslant C_{\alpha,a}\int_0^a \phi_f(u)^2\,\mathrm{d}u,\end{aligned}$$

which implies that $\mathscr{H}$ is continuously embedded in $L^2([0,a])$. Finally, from [15, Theorem 4.2], we have that the step functions are included in $\mathscr{H}$. Thus the proof is finished. $\square$

Now we introduce the linear operator $T : \mathscr{H} \subset \mathscr{H}_0 \to \mathscr{H}_0$, where $\mathscr{H}_0 = L^2([0,a])$, defined by

$$(Tf)(u) = C_H^{1/2} u^{\alpha} D^{\alpha}_{a-}(s^{-\alpha}f(s))(u) \tag{4.2}$$

$$= C_H^{1/2}\phi_f(u). \tag{4.3}$$

Henceforth, D^{α}_{0+} denotes the *left-sided fractional derivative*. It is the inverse operator of the left-sided fractional Riemann–Liouville integral of order α. This fractional integral is also defined for integrable functions f on $[0,a]$ and it is given by

$$I^{\alpha}_{0+}(f)(s) = \Gamma(\alpha)^{-1}\int_0^s f(r)(s-r)^{\alpha-1}\,\mathrm{d}r \quad \text{for a.a. } s \in [0,a].$$

Proposition 4.2. *Let $g : [0,a] \to \mathbb{R}$ be a function such that $u \mapsto u^{\alpha}g(u)$ belongs to $I^{\alpha}_{0+}(L^q([0,a]))$ for some $q > \alpha^{-1} \vee H^{-1}$. Then, $g \in \mathscr{D}(T^*)$ and for $u \in [0,a]$*

$$(T^*g)(u) = C_H^{1/2} u^{-\alpha} D^{\alpha}_{0+}(s^{\alpha}g(s))(u). \tag{4.4}$$

Proof. We first observe that the fact that $q > H^{-1}$ implies that the right-hand side of (4.4) is in $L^2([0,a])$. Secondly, $q > \alpha^{-1}$, [17, Corollary 2 of Theorem 2.4] and (4.2) imply

$$\int_0^a (Tf)(u)g(u)\,\mathrm{d}u = C_H^{1/2}\int_0^a u^{-\alpha}f(u)D_{0+}^{\alpha}(s^{\alpha}g(s))(u)\,\mathrm{d}u.$$

Therefore, $g \in \mathscr{D}(T^*)$ and (4.4) holds. $\square$

Proposition 4.3. *The operator T given by* (4.2) *satisfies conditions* (H1)–(H3).

Proof. Using the definition of the operator T (see (4.3)) and (4.1) we obtain for any $f \in \mathscr{H}$

$$|Tf|_{\mathscr{H}_0} = |Tf|_{L^2([0,a])} = C_H^{1/2}|\phi_f|_{L^2([0,a])} = |f|_{\mathscr{H}}$$

and (H1) holds. In order to check condition (H2), we define

$$\mathscr{H}_* = \{f \in \mathscr{H} : \exists f^* \in L^{\infty}([0,a]) \text{ such that } \phi_f(u) = u^{-\alpha}I_{0+}^{\alpha}(s^{\alpha}f^*(s))(u)\}.$$

Proposition 4.2 implies that for any $f \in \mathscr{H}_*$, ϕ_f belongs to $\mathscr{D}(T^*)$. Hence, $\mathscr{H}_* \subset \mathscr{T}_{\mathscr{H}}$. Therefore, in order to show (H2), it suffices to prove that $\mathscr{H}_*$ is a dense set of $\mathscr{H}$. By (4.1) we only need to show that the family

$$L_*^2 = \{f : [0,a] \to \mathbb{R} : \exists f^* \in L^{\infty}([0,a]) \text{ such that } f(u) = u^{-\alpha}I_{0+}^{\alpha}(s^{\alpha}f^*(s))(u)\}$$

is a dense subset of $L^2([0,a])$. For this, let $g \in L^2([0,a])$ be such that for any $f^* \in L^{\infty}([0,a])$,

$$0 = \int_0^a g(u)u^{-\alpha}I_{0+}^{\alpha}(s^{\alpha}f^*(s))(u)\,\mathrm{d}u.$$

Hence, by [17, equality (2.20)] we obtain

$$0 = \int_0^a I_{a-}^{\alpha}(s^{-\alpha}g(s))(u)u^{\alpha}f^*(u)\,\mathrm{d}u.$$

Consequently, $g = 0$ (see [17, Lemma 2.5]) because $L^{\infty}([0,a])$ is dense in $L^1([0,a])$, and therefore L_*^2 is dense in $L^2([0,a])$.

Finally, Proposition 4.2 implies that $L^{\infty}([0,a]) \subset \mathscr{T}_{L^2([0,a])}$, and (H3) holds. $\square$

Proposition 4.4. *B^H belongs to* $\mathrm{Dom}^*\delta$.

Proof. We know that $B_t^H = I_1(1_{[0,t]}) \in L^2(\Omega; L^2([0,a]))$. The symmetrization of $1_{[0,t]}$ as an element of $(L^2([0,a]))^{\otimes 2}$ is $\tilde{1}_{[0,t]}(\cdot) = \frac{1}{2}(1 \otimes 1)$. Hence, by Theorem 3.2 and the fact that step functions are dense in $\mathscr{H}$ (see [15, Theorem 4.2]) we get the result. $\square$

Proposition 4.5. *Let $H \in (0, \frac{1}{4})$. Then B^H is not in $\mathscr{H}$ w.p.*1.

Proof. Cheridito and Nualart [8] have proven that there is a sequence $(t_n)_n$ tending to zero such that

$$t_n^{-2H}\int_0^{a-t_n}(B_{s+t_n}^H - B_s^H)^2\,\mathrm{d}s \to a \quad \text{as } n \to \infty \text{ w.p.1.} \tag{4.5}$$

On the other hand, if there is $\omega_0 \in \Omega$ such that $B^H_\cdot(\omega_0) \in \mathscr{H}$, then [3, Proposition 6] and [17, property (6.40)] imply

$$\int_0^{a-t} (B^H_{t+s}(\omega_0) - B^H_s(\omega_0))^2 \, \mathrm{d}s = \mathrm{o}(t^{2\alpha}). \tag{4.6}$$

Finally, the result follows from the fact that both (4.5) and (4.6) cannot be satisfied at the same time since $H \in (0, \frac{1}{4})$. $\square$

The results of this section are proved by Cheridito and Nualart [8] in the case of a fBm on the whole real line, whereas here we consider a fBm on a finite interval $[0, a]$.

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Appendix A. The derivative operator D_T

Here we analyze the operator D_T presented in Section 2.2. Henceforth we suppose that conditions (H1)–(H3) hold.

Lemma A.1. *The operator D_T defined in* (2.3) *is closable from $L^2(\Omega; \mathscr{K})$ into $L^2(\Omega; \mathscr{H}_0 \otimes \mathscr{K})$.*

Proof. Let $(F_n)_n \subset \mathscr{S}_T(\mathscr{K})$ be a sequence that goes to zero in $L^2(\Omega; \mathscr{K})$ such that $D_T(F_n)$ converges to Y in $L^2(\Omega; \mathscr{H}_0 \otimes \mathscr{K})$. Then (H1) implies that for every $F \in \mathscr{S}_T(\mathbb{R})$, $h \in \mathscr{H}$ and $k \in \mathscr{K}$,

$$\begin{aligned} &E\langle Y, Fh \otimes k \rangle_{\mathscr{H}_0 \otimes \mathscr{K}} \\ &\quad = \lim_{n\to\infty} E\langle (T \otimes I_{\mathscr{K}})^*(T \otimes I_{\mathscr{K}}) DF_n, Fh \otimes k \rangle_{\mathscr{H}_0 \otimes \mathscr{K}} \\ &\quad = \lim_{n\to\infty} E\{\langle F_n, k \rangle_{\mathscr{K}} (FW(h) - \langle T^* TDF, h \rangle_{\mathscr{H}_0})\} \\ &\quad = 0 \end{aligned}$$

because $F_n \to 0$ in $L^2(\Omega; \mathscr{K})$. The second equality in the above equation follows from the integration by parts formula. Hence $Y = 0$ and the result holds. $\square$

Throughout, the closure of the operator D_T (with domain $\mathbb{D}^{1,2}_T(\mathscr{K})$ introduced in Section 2.2) will be also denoted by D_T.

Lemma A.2. *Let $F \in \mathbb{D}^{1,2}_T(\mathscr{K})$. Then $F \in \mathbb{D}^{1,2}(\mathscr{K})$.*

Proof. Let $F \in \mathscr{S}_T(\mathscr{K})$ be such that $F = Gk$, $G \in \mathscr{S}_T(\mathbb{R})$ and $k \in \mathscr{K}$. Then (H1) gives

$$\begin{aligned} |DF|^2_{\mathscr{H}\otimes\mathscr{K}} &= \langle DG \otimes k, DG \otimes k \rangle_{\mathscr{H}\otimes\mathscr{K}} \\ &= \langle DG \otimes k, T^*TDG \otimes k \rangle_{\mathscr{H}_0\otimes\mathscr{K}} \\ &\leqslant C|DF|_{\mathscr{H}\otimes\mathscr{K}} |D_T F|_{\mathscr{H}_0\otimes\mathscr{K}}, \end{aligned}$$

where we use that $\mathscr{H}$ is continuously embedded in $\mathscr{H}_0$. That is

$$|DF|_{\mathscr{H}\otimes\mathscr{K}} \leqslant C|D_T F|_{\mathscr{H}_0\otimes\mathscr{K}}. \tag{A.1}$$

Finally, the definitions of the spaces $\mathbb{D}^{1,2}_T(\mathscr{K})$ and $\mathbb{D}^{1,2}(\mathscr{K})$, and (A.1) imply that the proof is finished. □

Lemma A.3. *Let $F \in \mathbb{D}^{1,2}_T(\mathscr{K})$. Then $(T \otimes I_{\mathscr{K}})DF$ belongs to* $\mathrm{Dom}(T \otimes I_{\mathscr{K}})^*$ *w.p.*1. *Moreover*

$$D_T F = (T \otimes I_{\mathscr{K}})^*(T \otimes I_{\mathscr{K}})DF.$$

Proof. Let $(F_n)_n \subset \mathscr{S}_T(\mathscr{K})$ be a sequence that converges to F in $\mathbb{D}^{1,2}_T(\mathscr{K})$ as $n \to \infty$. Then, using (H1), we obtain

$$\begin{aligned} &E|(T \otimes I_{\mathscr{K}})(DF_n - DF)|^2_{\mathscr{H}_0\otimes\mathscr{K}} \\ &\quad = E|DF_n - DF|^2_{\mathscr{H}\otimes\mathscr{K}} \to 0 \quad \text{as } n \to \infty \end{aligned} \tag{A.2}$$

because of (A.1) and

$$E|(T \otimes I_{\mathscr{K}})^*(T \otimes I_{\mathscr{K}})DF_n - D_T F|^2_{\mathscr{H}_0\otimes\mathscr{K}} \to 0 \quad \text{as } n \to \infty. \tag{A.3}$$

Now the result is a consequence of (A.2) and (A.3) and the fact that $(T \otimes I_{\mathscr{K}})^*$ is a closed operator (see [16, Theorem VIII.1]). □

The proof of the following result uses standard arguments. So we only sketch it.

Lemma A.4. *Let $\phi \in C^\infty(\mathbb{R})$ be a function with compact support and $u \in \mathbb{D}^{1,2}_T(\mathscr{K})$. Then $\phi(|u|^2_{\mathscr{K}})$ belongs to $\mathbb{D}^{1,2}_T(\mathscr{K})$ and*

$$D_T(\phi(|u|^2_{\mathscr{K}})) = 2\phi'(|u|^2_{\mathscr{K}})((T \otimes I_{\mathscr{K}})^*(T \otimes I_{\mathscr{K}})Du)^*(u).$$

Proof. It is easy to see that the result holds for $u \in \mathscr{S}_T(\mathscr{K})$, using only the definition of D_T for smooth $\mathscr{K}$-valued random variables. Now let $(F_n)_n \subset \mathscr{S}_T(\mathscr{K})$ be a sequence such that

$$||F_n - u||_{1,2,T} \to 0 \quad \text{as } n \to \infty.$$

Hence

$$\begin{aligned} &|\phi'(|u|^2_{\mathscr{K}})((T \otimes I_{\mathscr{K}})^*(T \otimes I_{\mathscr{K}})Du)^*(u) \\ &\quad - \phi'(|F_n|^2_{\mathscr{K}})((T \otimes I_{\mathscr{K}})^*(T \otimes I_{\mathscr{K}})DF_n)^*(F_n)|_{\mathscr{H}_0} \end{aligned}$$

$$
\begin{aligned}
&\leqslant |\phi'(|u|^2_{\mathscr{K}})||u|_{\mathscr{K}}|(T\otimes I_{\mathscr{K}})^*(T\otimes I_{\mathscr{K}})(Du-DF_n)|_{\mathscr{H}_0\otimes\mathscr{K}}\\
&\quad+|\phi'(|u|^2_{\mathscr{K}})||(T\otimes I_{\mathscr{K}})^*(T\otimes I_{\mathscr{K}})DF_n|_{\mathscr{H}_0\otimes\mathscr{K}}|u-F_n|_{\mathscr{K}}\\
&\quad+|\phi'(|u_{\mathscr{K}}|^2)-\phi'(|F_n|^2_{\mathscr{K}})||(T\otimes I_{\mathscr{K}})^*(T\otimes I_{\mathscr{K}})DF_n|_{\mathscr{H}_0\otimes\mathscr{K}}|F_n|_{\mathscr{K}}\\
&\to 0 \quad \text{in probability.}
\end{aligned}
$$

Finally, the fact that D_T is a closed operator and

$$
\begin{aligned}
&|\phi'(|F_n|^2_{\mathscr{K}})||((T\otimes I_{\mathscr{K}})^*(T\otimes I_{\mathscr{K}})DF_n)^*(F_n)|_{\mathscr{H}_0}\\
&\quad\leqslant C|(T\otimes I_{\mathscr{K}})^*(T\otimes I_{\mathscr{K}})DF_n|_{\mathscr{H}_0}
\end{aligned}
$$

imply the result. □

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